Lecture #12

I. Orthogonality
II. Commutation
III. Eigenvalue Spectrum
IV. Linear Combinations of Degenerate Eigenfunctions = Eigenfunctions

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I. Orthogonality

ψ_n and ψ_m are eigenfunctions of hermitian operators then these are *orthogonal*: $\int \psi_n^* \psi_m d\tau=0$. Eigenvalues of hermitian operators are real.
Remember to use two different indicies n and m when evaluating orthogonality.
show that $\{\psi_n(x) = \exp(-i\theta)$ for $0<\theta<2\pi\}$

$$\int_0^{2\pi} \psi_m^* \psi_n d\theta = \int_0^{2\pi} e^{-im\theta} e^{in\theta} d\theta = \int_0^{2\pi} e^{-im\theta} e^{in\theta} d\theta = \int_0^{2\pi} e^{i(n-m)\theta} d\theta$$

$$\int_0^{2\pi} \cos(n-m) + i\sin(n-m) d\theta = \sin(n-m)\Big|_0^{2\pi} - i\cos(n-m)\Big|_0^{2\pi} = 0$$

A set of eigenfunctions both normalized and orthogonal constitute orthonormal basis set

$$\int \psi_{\mathbf{k}}^* \psi_{\mathbf{n}} d\tau = \delta_{\mathbf{nk}} \textbf{ kronecker delta = 1 or 0 : orthonormality}$$

II. Commutation

Two operators **A**,**B** are said to commute if the **AB**f(x)=**BA**f(x) which is equivalent to **AB**f(x)-**BA**f(x)= 0 →{**AB-BA**}f(x)=0. {**AB-BA**} = **O** where **0** is the multiply by zero operator. is commutator, written as **[A,B]**.
Thus if A,B, commute **[A,B]** = **0**
[**p**$_{\mathbf{x}}$,**x**] =

$$[p_x, x]\psi = p_x x\psi - xp_x\psi = -i\hbar\frac{\partial}{\partial x}\{x\psi\} - x\left\{-i\hbar\frac{\partial \psi}{\partial x}\right\} = -i\hbar\left\{1\psi + x\frac{\partial \psi}{\partial x}\right\} + i\hbar x\frac{\partial \psi}{\partial x}$$

$$= -i\hbar\psi - i\hbar x\frac{\partial \psi}{\partial x} + i\hbar x\frac{\partial \psi}{\partial x} = -i\hbar\psi = -i\hbar I\psi \qquad [p_x, x] = -i\hbar I \quad \textit{to show operators on both sides.}$$

Where **I** is the identity operator. Thus **p**$_{\mathbf{x}}$ and **x** don't commute.
For 2 noncommuting operators **A**,**B**: Δ**A**Δ**B**≥? |<C>| where [**A**,**B**] = iC, *If 2 operators do not commute we can't simultaneously know precise values for corresponding observables* thus for **p**$_{\mathbf{x}}$ & **x**: Δ**p**$_{\mathbf{x}}$Δ**x**≥h/4π, The Heisenberg uncertainty principle.

III. Eigenvalue Spectrum – The distribution of eigenvalues for an operator. If an operator **O** operates on Ψ it will return oΨ**,** even though Ψ may not be an eigenfunction of **O**.

$\mathbf{O}Y = oY$ o could be $o_1, o_2, o_3, o_4 \ldots$

Spectrum = {o} = { $o_1, o_2, o_3, o_4 \ldots$}

If Y is **not** an eignfunction of **O**, the o values are still eignvalues, we just do not know which eignvalues, i.e. a distribution. This means if you carried out the corresponding experiment for the **O** operating on Y you may get one eignvalue for o, but if you repeat it you obtain yet another value for o. Thus you obtain a distribution of the o eigenvalue spectrum, thus a certain probability of obtaining a particular value. If $Y_{\mathbf{n}}$ is an eigenstate of **O**, you will always obtain the same (precise) eigenvalue of o.

$$\mathbf{O}Y_{\mathbf{n}} = o_n Y_{\mathbf{n}}$$

Eigenvalues may be discrete or continuous. Imposition of boudary conditions on eigenfunctions leads to quantization of the eigenvalues (observables). Thus for the Schrodinger equation, imposing boundary condition on ψ leads to quantization of observable E.

IV. Linear combinations of degenerate eigenfunctions are also eigenfunctions

$\mathbf{H}\psi_n = E_n\psi_n$ has eigenfunctions $\psi_1,\psi_2,\psi_3\ldots$

For each eigenvalue (energy level) E_n, if there is only one eigenfunction (energy state, or state vector) $\rightarrow$ nondegenerate. If we have several eigenfuctions for a particular eigenvalue, these are said to be degenerate. This means:

$\mathbf{H}\psi_1 = E\psi_1$, $\mathbf{H}\psi_2 = E\psi_2$, $\mathbf{H}\psi_3 = E\psi_3$, and if $\phi_1 = c_{11}\psi_1+c_{12}\psi_2+c_{13}\psi_3$ then ϕ_1 is an eigenfunction of **H**.

Prove this-- simply operate on both sides of def with **H** and note that it is linear.

So other linear combinations such $\phi_2 = c_{21}\psi_1+c_{22}\psi_2+c_{23}\psi_3$, $\phi_3 = c_{31}\psi_1+c_{32}\psi_2+c_{33}\psi_3$ are eigenfuctions. Thus the system may be described by the set of eigenfunctions $\{\psi_1,\psi_2,\psi_3\}$ or $\{\phi_1, \phi_2, \phi_3\}$ if that they are linearly independent (linearly independent functions are such that no relation $a_1\phi_1+a_2\phi_2+a_3\phi_3=0$, no function can be expressed as a l.c. of the others). This is an example of the principle of superposition - If a state is equally well described by either of several functions (degenerate functions ψ_1,ψ_2,ψ_3) then it is equally well described by the linear combination of these functions (ϕ_1).